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Some comments on the connection between disordered long range spin glass models and their mean field version

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Abstract

In this paper, we consider a particular aspect of the relationship between mean field and finite-dimensional spin glasses. By means of a simple interpolation method, we prove that the free energy of a class of finite-dimensional spin glass models with Kac-type interactions is bounded below by that of their mean field analogue. As a result, Parisi theory of replica symmetry breaking can be exploited in order to give bounds on their free energy and ground state energy. Similar results hold for diluted versions of the systems.

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1. Introduction

The relationship between finite-range and mean field models is one of the most debated issues in the field of spin glass systems (see, for instance, [1–3] for different points of view in this controversy). In particular, one would like to understand whether, and to what extent, the picture one gains from the study of the mean field Sherrington–Kirkpatrick model [4] (e.g., ultrametric organization of the equilibrium states, replica symmetry breaking [5]) applies to the Edwards–Anderson [6] model, at least when the space dimensionality is sufficiently high. The scope of the present paper is more modest, and we study the relation between the free energy of the mean field and that of some finite-dimensional models, having in mind especially the case where the interaction range diverges. This is a very natural question to ask, which in the context of translation-invariant, non-disordered systems was studied a long time ago by Lebowitz and Penrose [7]. Lebowitz and Penrose [7] considered systems of particles interacting via a two-body potential of Kac type [8], whose interaction range can tend to infinity while the total strength remains constant. Under broad conditions, they proved that in this limit the infinite-volume free energy density approaches that corresponding to the mean field theory of van der Waals (or, in the case of spin systems, of Curie and Weiss).

Spin glass models with Kac-type interaction have already been considered in the literature [9–11]. In particular, it was proved that, at sufficiently high temperature and zero magnetic field, the free energy of the Sherrington–Kirkpatrick model coincides with that of its Kac-type analogue, in the infinite-range limit. However, we are not aware of more general results valid at low temperature and in the presence of an external field. In the present paper, we consider the Sherrington–Kirkpatrick and the diluted Viana–Bray [12] mean field models, and their Kac-type counterparts. Our main result is that, under some conditions on the interaction, the free energy of the mean field models is a lower bound for that of the corresponding finite-dimensional Kac-type one, at any β and h. In particular, this implies that 'broken replica symmetry bounds', such as those proved in [13, 14] for the mean field models, hold also for the finite-dimensional systems.

Our proof is based on very simple interpolation ideas, such as those employed for instance in [13, 15].

2. The Kac–SK model

Let us define the Kac–SK Hamiltonian introduced in [9, 10]. The model lives on the ν-dimensional lattice $\mathbb{Z}^\nu$, and the elementary degrees of freedom are Ising spins $\sigma_i = \pm 1$, $i \in \mathbb{Z}^\nu$. Given $\gamma > 0$ and any finite subset $\Lambda \subset \mathbb{Z}^\nu$ of cardinality $|\Lambda|$, we define the finite volume Hamiltonian as

$$H_\Lambda^{(\gamma)}(\sigma, h; J) = -\frac{1}{\sqrt{2W(\gamma)}} \sum_{i,j\in\Lambda} \sqrt{w(i-j;\gamma)} J_{ij}\sigma_i\sigma_j - h\sum_{i\in\Lambda}\sigma_i. \tag{1}$$

Here,

$$W(\gamma) = \sum_{i\in\mathbb{Z}^\nu} w(i;\gamma) \tag{2}$$

and

$$w(r;\gamma) = \gamma^\nu \phi(\gamma r) \tag{3}$$

for some smooth, non-negative and normalized real function ϕ:

$$\phi(r) \geqslant 0 \tag{4}$$

$$\int \phi(r)\,\mathrm{d}^\nu r = \int w(r;\gamma)\,\mathrm{d}^\nu r = 1. \tag{5}$$

For reasons of simplicity, we also require that

$$\phi(r) \leqslant C\,\mathrm{e}^{-C'|r|} \tag{6}$$

for some positive constants C, C'. The J_{ij} are i.i.d. Gaussian random variables with mean zero and unit variance. Note that J_{ij} is independent of J_{ji}, for $i \neq j$. The parameter γ represents the effective inverse interaction range, since the variance of the coupling $\sqrt{w(i-j;\gamma)}J_{ij}$ decays to zero over a distance of order γ^{-1}. Due to the smoothness of $\phi(r)$, one has

$$W(\gamma) = \int \phi(r)\,\mathrm{d}^\nu r + o(\gamma) = 1 + o(\gamma).$$

On the other hand, recall that the Hamiltonian of the Sherrington–Kirkpatrick model is defined as

$$H_\Lambda^{\mathrm{SK}}(\sigma, h; J) = -\frac{1}{\sqrt{2|\Lambda|}} \sum_{i,j\in\Lambda} J_{ij}\sigma_i\sigma_j - h\sum_{i\in\Lambda}\sigma_i. \tag{7}$$

Of course, the lattice structure is completely irrelevant in this case, and the reason why we write the Hamiltonian in form (7) is to make the comparison with the Kac–SK model more transparent.

As usual, for a given inverse temperature β one introduces the disorder-dependent partition function Z_Λ, the Boltzmann–Gibbs state $\omega_J(\cdot)$ and the quenched free energy density f_Λ as

$$Z_\Lambda^{(\gamma)}(\beta, h; J) = \sum_\sigma \mathrm{e}^{-\beta H^{(\gamma)}(\sigma,h;J)} \tag{8}$$

$$\omega_J(A) = \left(Z_\Lambda^{(\gamma)}\right)^{-1} \sum_\sigma A(\sigma)\,\mathrm{e}^{-\beta H^{(\gamma)}(\sigma,h;J)} \tag{9}$$

$$f_\Lambda^{(\gamma)}(\beta, h) = -\frac{1}{\beta|\Lambda|} E \ln Z_\Lambda^{(\gamma)}(\beta, h; J) \tag{10}$$

where A is a generic observable and E denotes expectation with respect to the disorder J. We are considering free boundary conditions for simplicity. Moreover, consider a generic number n of independent copies (replicas) of the system, characterized by the spin variables $\sigma_i^1, \sigma_i^2, \dots, \sigma_i^{(n)}$, distributed according to the product Boltzmann–Gibbs state

$$\Omega_J = \omega_J^1 \omega_J^2 \dots \omega_J^n \tag{11}$$

where each $\omega_J^a(\cdot)$ acts on the corresponding set of σ_i^a, and all replicas are subject to the *same* disorder realization. For a generic smooth function F of the configurations of the n replicas, we define the $\langle\cdot\rangle$ averages as

$$\langle F(\sigma^1, \dots, \sigma^n)\rangle = E\Omega_J(F(\sigma^1, \dots, \sigma^n)). \tag{12}$$

Similar definitions can be introduced for the Sherrington–Kirkpatrick model.

As is well known [16, 17] the infinite-volume limit

$$f^{(\gamma)}(\beta, h) = \lim_{\Lambda\uparrow\mathbb{Z}^\nu} f_\Lambda^{(\gamma)}(\beta, h) \tag{13}$$

exists, and is independent of the way Λ grows, provided that $\Lambda \uparrow \mathbb{Z}^\nu$ in the sense of van Hove [18]. In particular, we can choose Λ_L to be the ν-dimensional hypercube $\{-L, \dots, L\}^\nu$ of side $2L + 1$, with $L \in \mathbb{N}$, and let $L \to \infty$. On the other hand, the existence of the thermodynamic limit

$$f^{\mathrm{SK}}(\beta, h) = \lim_{|\Lambda|\to\infty} f_\Lambda^{\mathrm{SK}}(\beta, h) \tag{14}$$

for the Sherrington–Kirkpatrick model was proved in [15]. Therefore, it is very natural to compare the two infinite-volume free energy densities, especially in the limit $\gamma \to 0$, where the interaction range of the Kac–SK model diverges and the system is expected to resemble its mean field counterpart. To do so, we need to put a further restriction on the potential $w(r; \gamma)$. Given $k \in \Lambda_L$ and a function g defined on Λ_L, define the Fourier transform of g as

$$\tilde{g}_L(k) = \sum_{j\in\Lambda_L} g(j)\,\mathrm{e}^{\mathrm{i}\frac{2\pi}{2L+1}kj} \tag{15}$$

so that

$$g(j) = \frac{1}{|\Lambda_L|} \sum_{k\in\Lambda_L} \tilde{g}_L(k)\,\mathrm{e}^{-\mathrm{i}\frac{2\pi}{2L+1}kj} \qquad \forall j \in \Lambda_L. \tag{16}$$

Then, we require that

$$\tilde{w}_L(k; \gamma) \geqslant o(|\Lambda_L|^{-1}) \tag{17}$$

uniformly in k, i.e. $w(i-j;\gamma)$ is non-negative-definite, apart from a negligible error term (a similar condition was used in [7], section 3). For instance, one can easily check that this condition is satisfied by the potential originally introduced by Kac *et al* [8], and later used in [9] in the context of spin glasses, which corresponds to $\nu = 1$ and

$$w(r;\gamma) = \frac{\gamma}{2}\,\mathrm{e}^{-\gamma|r|}.$$

Theorem 1. *Assume that condition (17) holds. Then, for any β and h,*

$$f^{(\gamma)}(\beta, h) \geqslant f^{\mathrm{SK}}(\beta, h). \tag{18}$$

In particular, in the Kac limit one has

$$\liminf_{\gamma\to 0} f^{(\gamma)}(\beta, h) \geqslant f^{\mathrm{SK}}(\beta, h). \tag{19}$$

Of course, one would also like to prove that

$$\lim_{\gamma\to 0} f^{(\gamma)}(\beta, h) = f^{\mathrm{SK}}(\beta, h)$$

but this has not been possible so far (for partial results in this direction, in the case of high temperature and zero magnetic field, see [9, 10]).

In [13], it was shown that the Parisi solution, with an arbitrary number of steps of replica symmetry breaking, is a rigorous lower bound for the infinite-volume free energy of the Sherrington–Kirkpatrick model. Therefore, theorem 1 immediately implies

Corollary 1. *For any functional order parameter x [5, 13], one has*

$$f^{(\gamma)}(\beta, h) \geqslant -\beta^{-1}\bar{\alpha}(\beta, h; x) \tag{20}$$

where $\bar{\alpha}$ is the Parisi trial functional, as defined in [13].

In particular, letting $\beta \to \infty$, corollary 1 implies that the ground state energy density $-e_0^{(\gamma)}(h)$ of the Kac–SK model is bounded below by the ground state energy of the Parisi solution for the Sherrington–Kirkpatrick model [5]:

$$e_0^{(\gamma)}(h) \leqslant 0.7633\ldots. \tag{21}$$

Let us emphasize that, in contrast with the mean field case, inequalities (20) and (21) are not expected to give tight bounds for finite γ.

3. The diluted Kac–SK model

The Sherrington–Kirkpatrick model is a fully connected mean field spin glass, in the sense that every spin interacts with any other spin of the system, irrespective of their mutual distance. A lot of attention is presently devoted to diluted mean field spin glasses, especially for their connection with combinatorial optimization problems. In this case, a given spin interacts only with a finite number of other sites, even in the thermodynamic limit. The mean field character stems from the fact that these sites are chosen randomly among the $N-1$ possible ones.

The diluted version of the Sherrington–Kirkpatrick model was first introduced by Viana and Bray in [12]. For this work, it is convenient to define it in a way that differs slightly from the original one. Let α be a positive number (average connectivity) and let ξ_α^i, $i = 1, 2\ldots, |\Lambda|$, be i.i.d. Poisson random variables of mean value α. $\xi_\alpha^i \geqslant 0$ represents the (random) number of sites which interact with the spin σ_i. The locations of these sites, denoted by $j^i_{\mu_i}$, $\mu_i = 1, \ldots, \xi_\alpha^i$, are chosen independently and uniformly in $\{1, \ldots, |\Lambda|\}$. Finally, the values of the interaction couplings are i.i.d. centred random variables $J^i_{\mu_i}$. The resulting Hamiltonian is therefore

$$H_\Lambda^{\mathrm{VB}}(\sigma, h, \alpha; \mathcal{J}) = -\sum_{i\in\Lambda} \sigma_i \sum_{\mu_i=1}^{\xi_\alpha^i} J^i_{\mu_i} \sigma_{j^i_{\mu_i}} - h \sum_{i\in\Lambda} \sigma_i. \tag{22}$$

For simplicity, we assume the $J^i_{\mu_i}$ to be Bernoulli variables $J^i_\mu = \pm 1$, but more general situations can be considered. We denote by $\mathcal{J}$ the dependence of the Hamiltonian on the whole set of quenched disordered variables ξ^i_α, $j^i_{\mu_i}$, $J^i_{\mu_i}$. The existence of the thermodynamic limit for the quenched free energy density $f_\Lambda^{\mathrm{VB}}(\beta, h, \alpha)$ of the Viana–Bray model was proved by Franz and Leone in [14], by means of a smart interpolation method which exploits simple properties of Poisson random variables.

Keeping the above definitions in mind, it is natural to introduce the finite-range Kac analogue of the Viana–Bray model as follows. For any site i, extract a Poisson variable ξ^i_α. The ξ^i_α sites $j^i_{\mu_i}$ which interact with i are chosen in $\mathbb{Z}^\nu$ with weights

$$P\left(j^i_{\mu_i} = j\right) = \frac{w(j-i;\gamma)}{W(\gamma)}$$

where w is defined as in the previous section. An interaction $J^i_{\mu_i}$ is assigned to the couple $\left(i, j^i_{\mu_i}\right)$ only if $j^i_{\mu_i}$ falls inside Λ. The Hamiltonian of the model is therefore

$$H_\Lambda^{(\gamma)}(\sigma, h, \alpha; \mathcal{J}) = -\sum_{i\in\Lambda} \sigma_i \sum_{\mu_i=1}^{\xi_\alpha^i} \chi\left(j^i_{\mu_i}; \Lambda\right) J^i_{\mu_i} \sigma_{j^i_{\mu_i}} - h \sum_{i\in\Lambda} \sigma_i \tag{23}$$

where

$$\chi(j;\Lambda) = 1 \qquad \text{if} \quad j \in \Lambda \qquad \chi(j;\Lambda) = 0 \qquad \text{if} \quad j \notin \Lambda.$$

In analogy with theorem 1, one can prove the following:

Theorem 2. *If condition (17) holds, then for any β, h, α, one has*

$$f^{(\gamma)}(\beta, h, \alpha) \geqslant f^{\mathrm{VB}}(\beta, h, \alpha). \tag{24}$$

Also in this case, from the 'broken replica symmetry bounds' for the mean field diluted model [14], it follows that the Parisi solution is a lower bound for the infinite-volume free energy density of the finite-range model for any γ.

4. Proof of the results

Proof of theorem 1. The idea of the proof is to interpolate between the Kac–SK and the Sherrington–Kirkpatrick free energies. Then, assumption (17) will imply that the derivative with respect to the interpolating parameter has a non-negative sign, whence the statement of the theorem.

For $0 \leqslant t \leqslant 1$, define the auxiliary partition function

$$Z_\Lambda(t) = \sum_\sigma \exp\beta\left(\sqrt{\frac{t}{2}}\sum_{i,j\in\Lambda}\sqrt{\frac{w(i-j;\gamma)}{W(\gamma)}}J_{ij}\sigma_i\sigma_j + \sqrt{\frac{1-t}{2|\Lambda|}}\sum_{i,j\in\Lambda}J'_{ij}\sigma_i\sigma_j + h\sum_{i\in\Lambda}\sigma_i\right) \tag{25}$$

where J'_{ij} are independent copies of the random variables J_{ij}. To simplify notation, we write Λ instead of Λ_L. Of course, one has

$$-\frac{1}{\beta|\Lambda|}E\ln Z_\Lambda(1) = f_\Lambda^{(\gamma)}(\beta,h) \tag{26}$$

$$-\frac{1}{\beta|\Lambda|}E\ln Z_\Lambda(0) = f_\Lambda^{\mathrm{SK}}(\beta,h). \tag{27}$$

Employing integration by parts on the Gaussian disorder, for instance in [20], the t derivative is found to be

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\beta|\Lambda|}E\ln Z_\Lambda(t) = -\frac{\beta}{4}\left(\frac{1}{|\Lambda|}\sum_{i,j\in\Lambda}\frac{w(i-j;\gamma)}{W(\gamma)}\left(1-\langle\sigma_i^1\sigma_i^2\sigma_j^1\sigma_j^2\rangle\right)-\left(1-\langle q_{12}^2\rangle\right)\right) \tag{28}$$

where

$$q_{12}\equiv|\Lambda|^{-1}\sum_{i\in\Lambda}\sigma_i^1\sigma_i^2$$

denotes the overlap between the two configurations σ^1,σ^2. From the translation invariance of w and the fact that $W(\gamma)\leqslant\infty$ (see equations (2) and (6)), it follows [7] that:

$$\lim_{L\to\infty}\frac{1}{|\Lambda|}\sum_{i,j\in\Lambda}w(i-j;\gamma) = W(\gamma). \tag{29}$$

Now, let us deal with the remaining terms. Letting $\tau_i\equiv\sigma_i^1\sigma_i^2$, one has the following:

Lemma 1.

$$\frac{1}{|\Lambda|}\sum_{i,j\in\Lambda}w(i-j;\gamma)\langle\sigma_i^1\sigma_i^2\sigma_j^1\sigma_j^2\rangle = \frac{1}{|\Lambda|^2}\sum_{k\in\Lambda}\tilde{w}_\Lambda(k;\gamma)\langle|\tilde{\tau}_\Lambda(k)|^2\rangle+o(1) \tag{30}$$

where the term $o(1)$ vanishes in the thermodynamic limit.

Therefore, since

$$\lim_{L\to\infty}\frac{\tilde{w}_\Lambda(0;\gamma)}{W(\gamma)} = 1$$

and

$$\tilde{\tau}_\Lambda(0) = |\Lambda|q_{12}$$

one has

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\beta|\Lambda|}E\ln Z_\Lambda(t) = \frac{\beta}{4|\Lambda|^2}\sum_{k\in\Lambda,k\neq0}\frac{\tilde{w}_\Lambda(k;\gamma)}{W(\gamma)}\langle|\tilde{\tau}_\Lambda(k)|^2\rangle+o(1)\geqslant o(1) \tag{31}$$

where we employed condition (17) to estimate the error term. The statement of the theorem then immediately follows from integration on t between 0 and 1. □

Proof of lemma 1. Rewrite the lhs of equation (30) as

$$\frac{1}{|\Lambda|}\sum_{l\in\Lambda}{\sum_{j\in\Lambda}}'w(l-j;\gamma)\langle\tau_l\tau_j\rangle+O(L^{-1/2}) \tag{32}$$

where the sum is restricted to those sites j whose distance from the boundary of Λ_L is at least $\sqrt{L}$. Recalling the definitions (15) and (16), one can further express (32) as

$$\frac{1}{L^{3\nu}}{\sum_{j\in\Lambda}}'\sum_{k_1,k_2\in\Lambda}\exp\left(-\mathrm{i}\frac{2\pi}{2L+1}(k_1+k_2)j\right)\langle\tilde{\tau}_\Lambda(k_1)\tilde{\tau}_\Lambda(k_2)\rangle\tilde{w}_\Lambda(-k_1;\gamma)+O(L^{-1/2}) \tag{33}$$

where we used the fact that, thanks to the restriction on j and to condition (6), one has

$$\sum_{l\in\Lambda} w(l-j;\gamma)\exp\left(-\mathrm{i}\frac{2\pi}{2L+1}k_1(l-j)\right)=\tilde{w}_\Lambda(-k_1;\gamma)+O(\mathrm{e}^{-C\sqrt{L}}). \tag{34}$$

Finally, one removes the restriction on j in (33), producing another error term of order $L^{-1/2}$, and the statement of the lemma follows from summation over j, since $\tilde{\tau}_\Lambda(-k)=\tilde{\tau}^*_\Lambda(k)$. □

Remark. Note that, despite many similarities, there is an important difference between our approach and that of Lebowitz and Penrose [7], in comparing the free energy densities of the mean field model with that of the finite-range one. Indeed, their method is based on the fact that the interaction between two sites i, j is a slowly varying function of $i-j$, for small γ, whereas in our case it is only the *variance* of the interaction that satisfies this property, while the couplings themselves oscillate due to the random sign of J_{ij}. For this reason, the Lebowitz and Penrose [7] work directly on the Hamiltonian of the Kac model, showing that it can be approximated by the Curie–Weiss one, while we work on the disorder averaged internal energy, in order to use integration by parts on the disorder as in (28).

Proof of theorem 2. The proof is conceptually very similar to that of theorem 1 and we sketch only the main steps here. In order to keep the formulae readable, we restrict to the case $h=0$, the general situation presenting no additional difficulty.

The interpolating partition function $Z_\Lambda(t)$ takes, in this case, the form

$$Z_\Lambda(t)=\sum_\sigma \exp\left(-\beta H^{(\gamma)}_\Lambda(\sigma,h=0,\alpha t;\mathcal{J})-\beta H^{\mathrm{VB}}_\Lambda(\sigma,h=0,\alpha(1-t);\mathcal{J}')\right) \tag{35}$$

where the random variables $\mathcal{J}$ appearing in the Hamiltonian of the finite-range system are chosen to be independent of those ($\mathcal{J}'$) of the mean field one. Note that the mean value of the Poisson variables ξ^i in the first term has been modified from α into αt, and in the second term from α into $\alpha(1-t)$. Of course, the analogue of equations (26) and (27) holds at the boundary points $t=0,1$. In order to compute the derivative of the t-dependent free energy as in (28), we need the following simple property of the probability distribution of a Poisson random variable ξ_λ of mean value λ:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}P(\xi_\lambda=k)=-P(\xi_\lambda=k)+P(\xi_\lambda=k-1)(1-\delta_{k,0}) \qquad k=0,1,2,\ldots \tag{36}$$

which follows from

$$P(\xi_\lambda=k)=\mathrm{e}^{-\lambda}\frac{\lambda^k}{k!}. \tag{37}$$

This trick proved to be very useful already in [14] and [19]. Then, one finds (we refer for details to [19], where a similar computation is done)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{|\Lambda|}E\ln Z_\Lambda(t)&=\alpha\left(\ln\cosh\beta\left(\frac{1}{|\Lambda|}\sum_{i,j\in\Lambda}\frac{w(j-i;\gamma)}{W(\gamma)}-1\right)+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\tanh^{2n}\beta}{n}\right.\\&\quad\left.\times\left(\langle q^2_{1\ldots 2n}\rangle-\frac{1}{|\Lambda|}\sum_{i,j\in\Lambda}\frac{w(j-i;\gamma)}{W(\gamma)}\langle\sigma^1_i\ldots\sigma^{2n}_i\sigma^1_j\ldots\sigma^{2n}_j\rangle\right)\right)\end{aligned} \tag{38}$$

where $q_{1\ldots 2n}$ denotes the multi-overlap between $2n$ configurations:

$$q_{1\ldots 2n}=|\Lambda|^{-1}\sum_{i\in\Lambda}\sigma^1_i\ldots\sigma^{2n}_i. \tag{39}$$

The first term on the rhs of equation (38) vanishes in the thermodynamic limit, thanks to (29). At this point, one employs lemma 1 (where this time $\tau_i \equiv \sigma_i^1 \ldots \sigma_i^{2n}$) and the convergence of the series

$$\sum_{n=1}^{\infty} \frac{\tanh^{2n}\beta}{n}$$

for finite β, to deduce that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\beta|\Lambda|}E\ln Z_{\Lambda}(t) \geqslant o(1) \tag{40}$$

which concludes the proof. □

5. Conclusions

In this paper, we analysed a particular aspect of the relationship between finite-dimensional and mean field spin glasses, namely, the comparison between their respective free energies. In particular, we considered the (mean field) Sherrington–Kirkpatrick model and its diluted Viana–Bray version on one hand, and their finite-dimensional counterparts with Kac-type interactions on the other. Our result is that, under suitable conditions on the Kac potential, the infinite-volume free energy density of the Kac-type models is bounded below by that of the infinite-range ones. While this result does not attack the problem of the existence or the nature of a phase transition in finite-ranged spin glasses, it shows nonetheless that Parisi's theory of replica symmetry breaking can be employed to obtain some non-trivial results for realistic models, e.g., rigorous bounds for the ground state energy. It would be interesting to compare these to numerical estimates from Monte Carlo simulations.

A very natural question to ask is whether the free energy of the finite-range system converges to that of the mean field model in the Kac limit, i.e. when the interaction range diverges. While this is believed to be true on physical grounds, a rigorous mathematical proof seems to be quite difficult to find. Note that this result would immediately give a new and independent proof of the existence of the infinite-volume limit for the Sherrington–Kirkpatrick free energy, a problem which has remained open for a very long time and has been solved only recently in [15].

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